Experimental observation of a periodic rotating wave in rings of unidirectionally coupled analog Lorenz oscillators

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In the present work we report on the experimental observation of a recently predicted behavior of coupled chaotic Lorenz oscillators that consists in a transition to periodic rotating waves of a shorter timescale than the uncoupled system. The experiment has been performed by designing an analog circuit corresponding to Lorenz differential equations, which is then used in the construction of an array of circuits with the appropriate coupling. [S1063-651X(98)10005-3]

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A number of recent studies have concentrated on the study and analysis of extended systems in far-from-equilibrium conditions that undergo pattern formation [1]. So far most of the extended systems that have been analyzed include fluid systems, lasers, chemical systems, etc., that are studied by using a continuum description, which is then simplified to a suitable amplitude equation (or analogous type of representation [1]). Instead, in the present work we shall consider discretely coupled (chaotic) systems that undergo also pattern formation from a spatially homogeneous initial state that is temporally chaotic.

Pattern formation in this kind of system was already considered long ago by Turing [2] in a seminal contribution. These types of systems are important in a biological context because in this situation, differing from the examples that arise when studying inanimate matter, the systems are intrinsically discrete, e.g., cells. In our work we shall consider coupled oscillators in a ring geometry, which has several interesting applications in a number of biological systems [3], especially when dealing with the so-called central pattern generators CPGs, which have been shown to play important roles in peripheral neural systems, locomotion, etc. [3,4]. In such cases the relevant interesting behavior arises as the system performs transitions from certain patterns (e.g., gaits) to different patterns as a relevant parameter in the system changes.

In the present work we shall report the experimental observation of the appearance of periodic discrete rotating waves in rings of unidirectionally coupled analog Lorenz oscillators. This behavior was already predicted theoretically for these systems in Ref. [5]. In that work it was argued that the transition to the periodic waves occurs as the system undergoes a symmetric Hopf bifurcation [3], i.e., a Hopf bifurcation in which the simultaneous crossing through the imaginary axis of a pair of complex conjugate eigenvalues does not take place as a parameter corresponding to single Lorenz oscillator is varied, but instead due to the symmetry introduced in the system by the coupling. In Ref. [5] the problem was analyzed by a performing a discrete Fourier transform of the linearized approximation to the system, which yields a problem in which a number of symmetry-related, namely, complex conjugate in pairs, modes coexist.

A feature of the discrete rotating waves that we are considering here that makes them very attractive potentially is that they are *faster* than the behavior corresponding to the uncoupled oscillators. As the behavior of the latter is aperiodic (chaotic), it is more correct to say that the time scale of the waves is considerably shorter than the time scale of the unperturbed chaotic oscillators. This feature is very attractive in the context of CPGs as it offers a dynamical intrinsic mechanism to generate a fast frequency in the system. This mechanism also might be useful in the design of artificial systems in which one wishes to incorporate this feature of a fast frequency in the system.

Analog circuits constitute a well-known [6,7] approach in the study of the behavior and characterization of nonlinear dynamical systems that are not solvable analytically and that appear in a variety of fields. An interesting feature of the simulation through analog circuits is the fact that they are real physical systems, implying that if a given phenomenon appears in the simulation, it must be a genuine behavior of the underlying equations, implying that it is not sensitive to noise, small parameter mismatch, etc. In addition, in analog circuits one can play with the time scales of the system to make the phenomenon under study faster or slower than the real one, which may offer advantages in many cases. Perhaps one of the most attractive features of these systems is related to the speed at which one may get significant results (compared to digital simulation, for which longer times are usually required). Another interesting feature of analog simulations that is linked to its speed is that large volumes of parameter space can be quickly surveyed for interesting phenomena, often by turning knobs to adjust the relevant parameters while examining the results on a visual display. Instead, the equivalent procedure with digital simulation would be very costly.

In our study we have studied the reported phenomena on arrays of Lorenz circuits by relying on an analog implemen-

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tation of the Lorenz [8] equations described below. The present implementation uses op amps as the building blocks for the operations of sum, difference, and integration and analog multipliers for the product terms. The variables x, y, and z will be the voltages at the output of the (Miller) integrators, where one has to take into account that analog circuits operate in a dynamical range -15 to 15 V, while analog multipliers operate in the more restricted range -10 to 10 V. Thus one needs to scale the original variables to ensure that all the voltages are inside this latter range. It is important to mention that the voltages at any point of the circuits (e.g., after multiplying a voltage by a constant), and not just across the capacitors, should be inside the -10 to 10 V range.

From the original set of equations

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= Rx - y - xz, \\ \dot{z} &= xy - bz, \end{aligned} \tag{1}$$

we have considered the transformations $u=x/5$, $v=y/5$, and $w=z/10$, which lead to the following form for the Lorenz equations:

$$\begin{aligned} \dot{u} &= \sigma(v-u), \\ \dot{v} &= Ru - v - 10\,uw, \\ \dot{w} &= 2.5\,uv - bw. \end{aligned} \tag{2}$$

These equations have been implemented in the circuit presented in Fig. 1. As can be seen from the schematic, we have employed Texas Instruments TL084-type op amps, while the multipliers are Analog Devices AD633. At this point it is important to notice that analog multipliers exhibit a large offset voltage at the output and it is necessary to introduce a compensation network to suppress it. The network used in our circuit is the one recommended by the manufacturer [9] and the circuits are manually adjusted to ensure the elimination of this offset. The values of the resistors are $R_1 = R_2 = R_3 = R_4 = R_6 = R_8 = R_{11} = R_{13} = 100\ \mathrm{k}\Omega$, $R_9 = 1\ \mathrm{M}\Omega$, $R_{10} = 10\ \mathrm{k}\Omega$, and $R_{12} = 250\ \mathrm{k}\Omega$, and $C_1 = C_2 = C_3 = 1$ nF. The parameters of the model, σ, R, and b, can be adjusted by varying R_5, R_7, and R_{14} in the form, $\sigma = 10^6\Omega/R_5$, $R = 10^{-5}R_7\Omega^{-1}$, and $b = 10^6\Omega/R_{14}$. The capacitors take the values $C_1 = C_2 = C_3 = 1$ nF. The tolerances of the resistors and capacitors are all 1%. All the experimental results have been measured by using a Hewlett-Packard 54600A digital oscilloscope with 20 MS/s and a record length of 4000 points. The suggested circuit exhibits the characteristic behavior of the Lorenz system, as can be seen from Fig. 2. Notice that the present implementation differs from that introduced by Cuomo *et al.* [10,11].

The main result of the present contribution consists in the experimental confirmation of the previous theoretical prediction that a fast periodic discrete rotating wave appears in rings of unidirectionally coupled Lorenz systems in the chaotic state. To obtain such a result one needs to build an array according to the equations [see Eq. (3) of Ref. [5]]

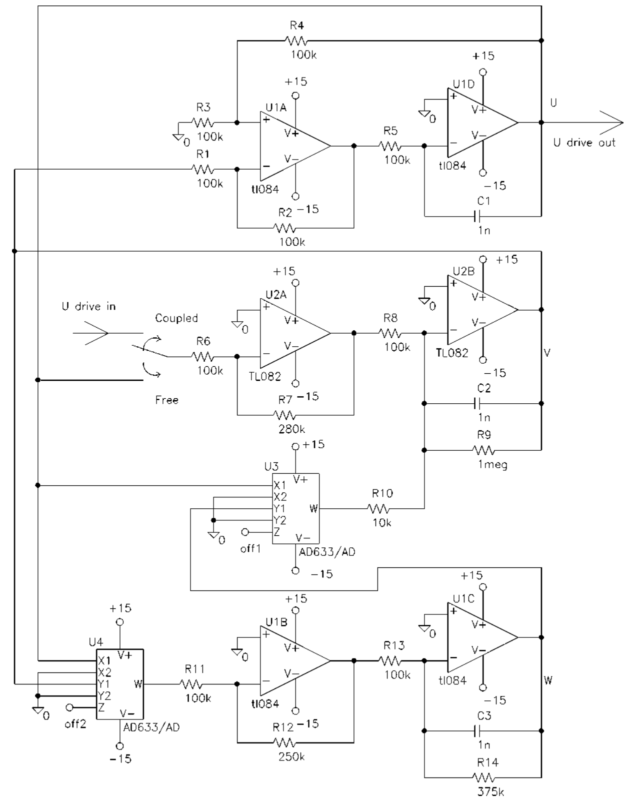

FIG. 1. Schematic of the Lorenz-based chaotic circuit, including the coupling with neighboring circuits. When the switch is in the *free* position the circuit is isolated. In the *coupled* position the Ru term is generated from the u circuit in the previous circuit. See the text for further details.

$$\begin{aligned} \dot{u}_j &= \sigma(v_j - u_j), \\ \dot{v}_j &= R\overline{u_j} - v_j - u_j w_j, \\ \dot{w}_j &= u_j v_j - bw_j \end{aligned} \tag{3}$$

for $j=1,\ldots,N$, where $\overline{u_j} = u_{j-1}$ ($\overline{u_1} = u_N$ as we are considering periodic boundary conditions). This coupling among the circuits, which was introduced in Ref. [12], has been implemented as shown in Fig. 1, i.e., each circuit is driven by its predecessor and simultaneously drives its successor through the Ru term.

The above-mentioned waves can be seen better from Fig. 3, where the waves corresponding to two Lorenz analog circuits in a ring with $N=3$ are plotted versus time [in Fig. 3(a)]. This plot shows the periodic wave form of the variables and also the $2\pi/3$ phase difference between neighboring oscillators that is characteristic of these waves [5]. This feature is also clear from Fig. 3(b), where these two waves are represented one versus the other. Moreover, if one compares the time scales of Figs. 2(a) and 3(a), then the fact that the wave is *fast* compared to the uncoupled system becomes clear.

In conclusion, we have presented experimental evidence of the periodic rotating waves that in Ref. [5] were predicted to appear in rings of Lorenz systems coupled in a certain way. These waves arise from an instability in the chaotic uniform synchronized state at a certain wavelength (i.e., size of the ring, as the system is discrete), which due to the par-

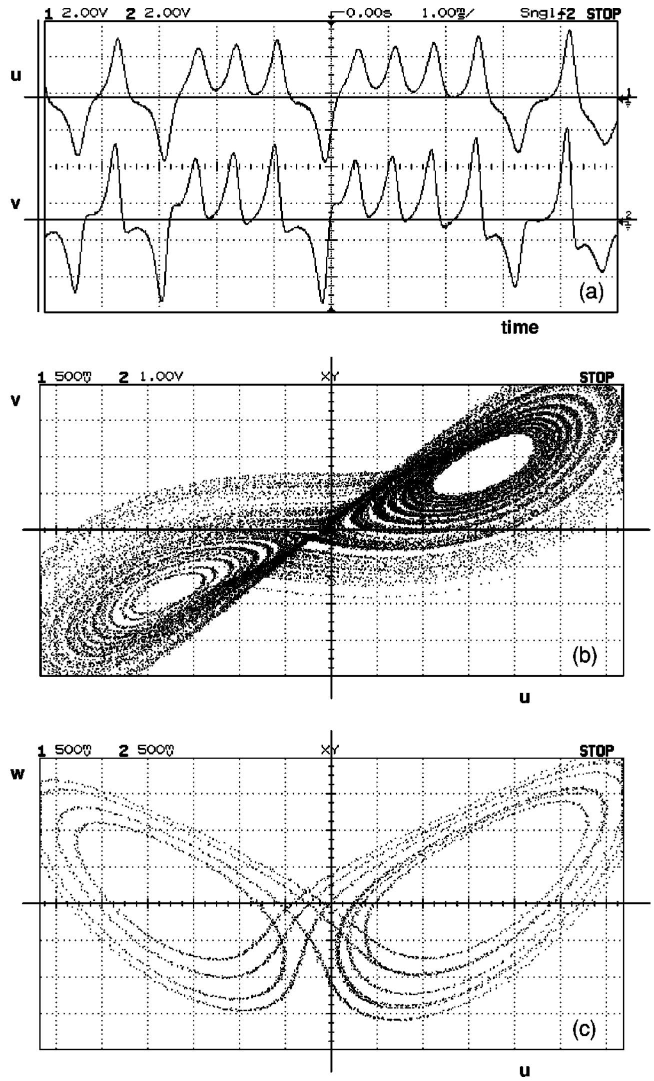

FIG. 2. Circuit data for an uncoupled Lorenz circuit: (a) u and v versus time, (b) phase portrait v vs u, and (c) phase portrait w vs u. Notice that more points have been taken in (b) than in (c), with the aim of facilitating the identification with the well-known features of the Lorenz attractor.

ticular arrangement of the system appears through a symmetric Hopf bifurcation. To our knowledge, this is the first evidence for a dynamical mechanism originating a faster time scale in a coupled dynamical system. Thus the observed be-

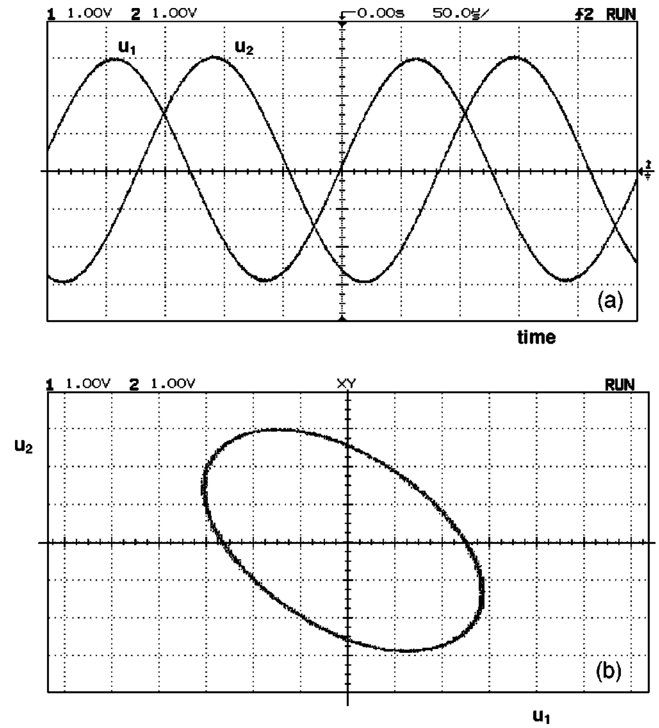

FIG. 3. Circuit data for a ring of three Lorenz circuits: (a) u for two contiguous circuits (u_1 and u_2) versus time and (b) phase portrait u_2 vs u_1.

havior is emergent, in the sense that it is not contained in the isolated systems, but appears once that they are coupled. Analogously, it disappears if the ring arrangement is broken. One is tempted to speculate about the possible implications in other fields of this type of collective behavior, i.e., in biology, where rings of neurons, namely, central pattern generators [3], could have fast dynamical time scales that could be behind some instances in which the system appears to be responding very fast.

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